Chaos via the Break-up of a Torus in a Semiconductor Laser with Phase-Conjugate Feedback

Bernd Krauskopf and Kirk Green

Department of Engineering Mathematics, University of Bristol, Bristol BS8 1TR, UK.
B.Krauskopf@bristol.ac.uk

Abstract: We present a detailed study of a route to chaos via quasiperiodicity involving tori in the system. More concretely, we show how chaos arises when a torus breaks up. This is made possible by computing what are called unstable manifolds of saddle periodic orbits, for which we developed a new numerical method.

Introduction

Possible applications of a semiconductor laser subject to phase-conjugate feedback (PCF) include mode locking, phase locking, and frequency control. This explains the considerable recent interest in the nonlinear dynamics of PCF lasers, which are known to exhibit a wealth of dynamics, including stable periodic operation, quasiperiodic motion and chaos [1,2]. Transitions to chaos were recently studied with a combination of bifurcation diagrams and phase plots [2]. The PCF laser is a delay differential system with an infinite dimensional phase space [2,3]. With new methods developed specifically for delay equations we are now able to study its dynamics in unprecedented detail.

Rate Equation Model

A single-mode semiconductor laser subject to weak (instantaneous) PCF is described by the rate equations [2]

$$\dot{E} = \frac{1}{2}\left[-i\alpha G_N(N(t)-N_{sol}) + (G - \frac{1}{\tau_p})\right]E(t)$$

$$+ \kappa E^*(t-\tau)\exp(2i\delta(t-\tau/2)$$

$$\dot{N} = \frac{I}{q} - \frac{N(t)}{\tau_e} - G|E(t)|^2$$

for the complex electric field E and for the inversion N. As the main parameter we study the dimensionless product $\kappa\tau$ of the feedback rate κ and the external-cavity round trip time τ. All other parameters are set to realistic values as in Ref. [2].

These rate equations describe how the history (or initial condition), defined on the interval $[-\tau,0]$ with values in the three-dimensional (E,N)-space, evolves in time. As is common, we consider the time-evolution in the three-dimensional (E,N)-space, but it is important to keep in mind that this is a projection of the system's infinite-dimensional dynamics. The rate equations are symmetric with respect to the transformation $E \to -E$, which is physically a phase shift by π. As a consequence, any attractor is either symmetric or it has a symmetric counterpart [3].

Bifurcation Diagram

As $\kappa\tau$ is changed the dynamics of the PCF laser produces stable periodic output with \`bubbles\` of more complicated dynamics in between. While some routes to chaos are like for low-dimensional systems, the real challenge is to study transitions to chaos that are due to the system's infinite-dimensional character.

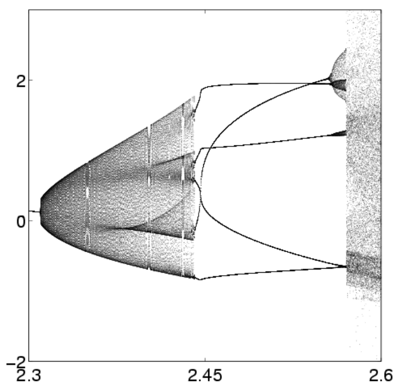

Figure 1: Bifurcation Diagram of normalised N (vertical) versus feedback parameter κτ (horizontal).

As a step in this direction, we consider in detail a transition to chaos at the beginning of one such bubble; see the bifurcation diagram (obtained by numerical integration) in Fig. 1. For each $\kappa\tau$ we allowed the system to settle down to an attractor and then plotted a normalised value of N when the trajectory crosses the value of average power in the positive direction. The single point for $\kappa\tau < 2.307$ corresponds to a stable periodic orbit. At $\kappa\tau = 2.307$ quasiperiodic dynamics emerge, which take place on an attracting torus. At $\kappa\tau = 2.440$ the dynamics on the torus become locked to a stable periodic orbit, which shows up as five branches. Physically the output is no longer quasiperiodic. The new stable periodic orbit itself develops quasiperiodic modulations at $\kappa\tau = 2.556$, a feature not found earlier in this system. Then at $\kappa\tau = 2.571$ the associated new torus suddenly disappears and the dynamics become chaotic. We note that there is no hysteresis in this sudden transition to chaos.

Manifold Computations

The key to understanding this transition lies in what happens to the torus after locking. For this it is not sufficient to use mere numerical simulation, because for $2.440 < \kappa\tau < 2.555$ one will only get an image of the stable periodic orbit, and not of the torus (or its remainder) on which it lies.

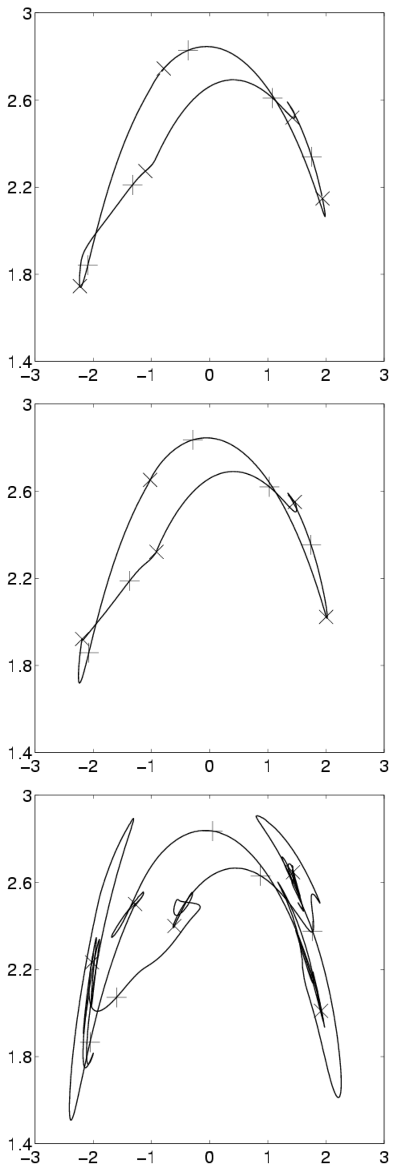

Figure 2: Break-up of torus for $\kappa\tau = 2.445$ (top), $\kappa\tau = 2.450$ (middle), and $\kappa\tau = 2.500$ (bottom).

We developed a numerical method (which will be discussed in detail elsewhere) for computing the unstable manifold of the saddle periodic orbit on the torus in a suitable Poincaré map. The saddle-point orbits themselves were found with the program DDE-Biftool [4]. In Fig. 2 we show plots for the values $\kappa\tau = 2.445$, $\kappa\tau = 2.450$ and $\kappa\tau = 2.500$ in the Poincaré section defined by $N = N_{avg} = 7.620 \times 10^8$. In each plot the crosses (+) mark the five intersection points of a saddle-type periodic orbit on the torus with the section. From each saddle-point there emanate two branches of the unstable manifold, which converge to two neighbouring attracting points (x). These five intersection points correspond to the five branches in Fig. 1. In this way, the torus, or what remains of it, shows as a closed curve.

Our computations show that the torus is smooth at $\kappa\tau = 2.445$, as is to be expected just after locking. However, it loses its smoothness by starting to `curl up` along the stable periodic orbit. This transition has just happened at $\kappa\tau = 2.450$. Physically, this corresponds to damped modulations of the periodic laser output. The torus becomes increasingly folded and stretched as $\kappa\tau$ increases. Nevertheless, the torus is still present as a continuous object, as evidenced by the continuous (but not smooth) closed curve at $\kappa\tau = 2.500$. (Notice that the different branches of the unstable manifold are allowed to intersect each other because we are looking at a two-dimensional projection of an infinite-dimensional system). Finally, in what appears to be an attractor crisis, chaotic dynamics are born at $\kappa\tau = 2.571$. The shape of the ensuing chaotic attractor is that of the unstable manifold of the saddle-type periodic orbit just before the crisis.

Conclusion

In summary, we found quasiperiodicity and the subsequent break-up of a torus in the PCF laser, which then led to chaos via an attractor crisis. This detailed study demonstrates the potential of our new method for computing unstable manifolds in delay systems. In future work, we will investigate in detail other routes to chaos that appear to be due to the infinite-dimensional nature of the PCF laser.

References

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